

Some Aspects of the Asymptotics Leading from Gas-Particles Equations Towards Multiphase Flows Equations

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Abstract This paper is devoted to the establishment at the formal level of a scaling leading from gas-particles (also called Eulerian-Lagrangian) models for monodisperse thick sprays towards multiphase flows models (also called Eulerian-Eulerian) including the volume fraction as an unknown. This passage to the limit involves the study of an inelastic collision operator acting on internal energies as well as velocities.

Keywords Sprays · Inelastic collisions · Multiphase flows · Gas-particles simulations

1 Introduction

Sprays are flows involving a continuous gaseous phase and a disperse phase (typically constituted of liquid droplets) whose volume fraction is not too big. We consider in this paper only monodisperse sprays (that is, all the droplets in the disperse phase have the same radius r). Moreover, we shall also suppose that all droplets are incompressible and that no evaporation occurs, so that r will be in the sequel an absolute constant.

We denote by $\alpha := \alpha(t, x) \in [0, 1]$ the volume fraction of gas at time $t \in \mathbb{R}_+$ and point $x \in \Omega$ (Ω being a subset of \mathbb{R}^3). Considering this quantity makes sense when the volume $\frac{4}{3}\pi r^3$ of a typical droplet is much smaller than a small (but macroscopic) elementary volume of fluid. We say that the spray is thick (it was first introduced in [16] and then used in the KIVA code [2, 3, 32, 33]) when $1 - \alpha(t, x)$ is not negligible in at least part of $\mathbb{R}_+ \times \Omega$ (typically $1 - \alpha(t, x) \gg 10^{-3}$) but not too big either (typically, $1 - \alpha(t, x) \leq 0.2$ at worst). We refer to [16, 32] for the concept of thick sprays.

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Thick sprays are modeled by a coupling of a kinetic equation and a fluid equation. This coupling is done through the volume fraction α and the drag between the two phases. We write below the set of equations described in [16], with a few differences that we explain in the sequel.

We denote by $\rho_g := \rho_g(t, x) \in \mathbb{R}_+$, $p := p(t, x) \in \mathbb{R}_+$, $u_g := u_g(t, x) \in \mathbb{R}^3$, $e_g := e_g(t, x) \in \mathbb{R}_+$, $E_g := E_g(t, x) = e_g(t, x) + \frac{1}{2}|u_g(t, x)|^2 \in \mathbb{R}_+$ and $T_g := T_g(t, x) \in \mathbb{R}_+$ the respective density (of mass), pressure, velocity, internal energy (per unit of mass), total (internal + kinetic) energy (per unit of mass), and temperature of the gas. Those quantities satisfy the following balance laws:

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \tag{1}$$

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \nabla_x p = -A, \tag{2}$$

$$\partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t\alpha = -B_1 - B_2, \tag{3}$$

where A is the momentum transferred to the (elementary volume at time t and point x of) gas by the dispersed phase and B_1, B_2 constitute the corresponding (resp. mechanical and thermal) transfer.

The density in the phase space $f := f(t, x, u_p, e_p) \geq 0$ of droplets which at time t and point x have velocity $u_p \in \mathbb{R}^3$ and internal energy $e_p \in \mathbb{R}_+$ satisfies the following Vlasov-Boltzmann equation:

$$\partial_t f + u_p \cdot \nabla_x f + \nabla_{u_p} \cdot (f\Gamma) + \partial_{e_p}(f\phi) = Q(f, f), \tag{4}$$

where Γ and ϕ represent the transfer of momentum and energy of the gaseous phase on a given droplet (which at time t and point x has velocity $u_p \in \mathbb{R}^3$ and internal energy $e_p \in \mathbb{R}_+$). Accordingly,

$$m_p \Gamma = -\frac{m_p}{\rho_p} \nabla_x p - D(u_p - u_g); \quad m_p \phi = \Phi(T_g - T_p), \tag{5}$$

$$A = \iint_{u_p, e_p} m_p \Gamma f \, du_p de_p, \tag{6}$$

$$B_1 = \iint_{u_p, e_p} m_p \left(\Gamma + \frac{\nabla_x p}{\rho_p} \right) \cdot u_p f \, du_p de_p, \tag{7}$$

$$B_2 = \iint_{u_p, e_p} m_p \phi f \, du_p de_p, \tag{8}$$

where m_p is the mass of one droplet, ρ_p is the density of the liquid constituting the droplets ($m_p = \frac{4}{3}\pi r^3 \rho_p$, and m_p, ρ_p, r are absolute constants), and T_p is the temperature of the droplet. In (5), the term $D(u_p - u_g)$ models the drag. The drag coefficient D is in general a function of $\rho_g, |u_g - u_p|$ (and also r, ρ_p and the molecular viscosity of the gas [this last quantity being neglected in the equation of momentum of the gas]).

Also in (5), the term $\Phi(T_g - T_p)$ models the thermal exchanges between the droplets and the gas. The coefficient Φ in general depends upon the thermal viscosity of the particle and the Nusselt number (and therefore upon $r, |u_g - u_p|$, etc.).

The system is closed thanks to the constitutive equations of the gas and the liquid:

$$p(t, x) = P_1(\rho_g(t, x), e_g(t, x)), \quad T_g(t, x) = T_1(\rho_g(t, x), e_g(t, x)), \tag{9}$$

$$T_p = T_2(e_p), \tag{10}$$

and the identity for the volume fraction of droplets:

$$1 - \alpha(t, x) = \frac{4}{3}\pi r^3 \iint_{u_p, e_p} f(t, x, u_p, e_p) du_p de_p. \tag{11}$$

The set of equations (1)–(11) is sometimes called ‘‘Gas-particles’’ or ‘‘Eulerian-Lagrangian’’. The main differences with the model proposed by Dukowicz [16] is that we take into account collisions (they were neglected in the original model) and equations for the energy [6].

Note that the presence of a non-infinitesimal volume fraction $1 - \alpha$ of droplets is not compatible with the presence of a non-infinite Boltzmann kernel (this is a consequence of the Boltzmann-Grad asymptotics: cf. [12]). The situation in the classical work of Dukowicz [16] is even worse since no collision kernel is considered there. The scaling that we propose in next section partially removes this incompatibility, since the collision kernel tends to infinity. The formulas for Q that we propose are original and thoroughly described in Sect. 2.1.

We provide in this work a link between (1)–(11) and a different class of systems, sometimes called ‘‘Eulerian-Eulerian’’, which models two-phase flows (including thick sprays). Those systems are thoroughly described in [24]. They are obtained at a heuristic level by taking averages of Euler-type equations for both phases, and by imposing reasonable closures.

In the ‘‘Eulerian-Eulerian’’ approach, the phase space density f of droplets is replaced by macroscopic quantities, namely: the density (of mass) $\rho := \rho(t, x) \in \mathbb{R}_+$ of liquid, its velocity $v := v(t, x) \in \mathbb{R}^3$, its internal energy (per unit of mass) $e := e(t, x) \in \mathbb{R}_+$, its total (internal + kinetic) energy (per unit of mass) $E := E(t, x) = e(t, x) + \frac{1}{2} |v(t, x)|^2 \in \mathbb{R}_+$ and its temperature $T := T(t, x) \in \mathbb{R}_+$. The equations write

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \tag{12}$$

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \alpha \nabla_x p = -\tilde{A}, \tag{13}$$

$$\partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p \partial_t \alpha = -\tilde{B}_1 - \tilde{B}_2, \tag{14}$$

$$\partial_t((1 - \alpha)\rho) + \nabla_x \cdot ((1 - \alpha)\rho v) = 0, \tag{15}$$

$$\partial_t((1 - \alpha)\rho v) + \nabla_x \cdot ((1 - \alpha)\rho v \otimes v) + (1 - \alpha)\nabla_x p = \tilde{A}, \tag{16}$$

$$\partial_t((1 - \alpha)\rho E) + \nabla_x \cdot \left((1 - \alpha)\rho \left(E + \frac{p}{\rho} \right) v \right) + p \partial_t(1 - \alpha) = \tilde{B}_1 + \tilde{B}_2. \tag{17}$$

Those balance laws are completed by the constitutive equations of the gas (similar to (9))

$$p(t, x) = P_1(\rho_g(t, x), e_g(t, x)); \quad T_g(t, x) = T_1(\rho_g(t, x), e_g(t, x)), \quad (18)$$

together with the constitutive equations of the liquid (incompressible) phase

$$T(t, x) = T_2(e(t, x)), \quad \rho(t, x) = \rho_p. \quad (19)$$

Finally, the transfer terms \tilde{A} , \tilde{B}_1 , \tilde{B}_2 of momentum and energy write

$$\tilde{A} = -(1 - \alpha) \frac{\rho}{m_p} \tilde{D}(v - u_g), \quad \tilde{B}_1 = -(1 - \alpha) \frac{\rho}{m_p} \tilde{D}(v - u_g) \cdot v, \quad (20)$$

$$\tilde{B}_2 = -(1 - \alpha) \frac{\rho}{m_p} \tilde{\Phi}(T - T_g). \quad (21)$$

The terms \tilde{A} , \tilde{B}_1 , \tilde{B}_2 respectively represent the drag force term, its deposit in terms of energy, and the thermal exchanges. The constants \tilde{D} , $\tilde{\Phi}$ respectively represent the drag force coefficient and the thermal conduction coefficient. They can be fitted using experimental data and in general depend upon α , $|v - u_g|$, etc. Note that systems like (12)–(19) appear not only in the theory of sprays, but also in many other kinds of multiphase flows (stratified, churning flows, etc.), the transfer terms (like \tilde{A} , etc.) depend in general of the type of flows which are considered and are generally obtained by using statistical averages [3, 24, 32, 33].

Our goal in this paper is to provide a clear scaling which enables to derive “rigorously at the formal level” macroscopic equations such as (12)–(21) from “gas-particles” equations such as (1)–(11). It is clear that (15)–(17) will be obtained by taking moments (with respect to v , e) of (4). This strategy has already been used in many works concerning the modeling of sprays [15, 27, 28], in the more complicated case when the spray is polydisperse: it uses however heuristic closures in order to derive the “Eulerian-Eulerian” equations.

Our approach, though it is restricted to the simpler case of monodisperse sprays, is quite different since:

- (i) It is based on a scaling of the sprays equation obtained after a non-dimensionalization of those equations;
- (ii) It provides *non heuristical* closures (that is, a mathematical link between A , B_1 , B_2 and \tilde{A} , \tilde{B}_1 , \tilde{B}_2);
- (iii) It involves the description of a new variant of the Boltzmann kernel where all the parameters are assessed.

In the scaling that we propose, the collision term Q appearing in (4) must be dominant. This exactly corresponds in the context of standard kinetic theory to the limit of small Knudsen number, in which $\frac{1}{\epsilon}$ is put in front of the collision kernel, and which leads from the Boltzmann equation of rarefied gases towards the compressible Euler equations of fluid dynamics (cf. [26] for a rigorous proof in the context of very smooth solutions on a small time interval, and [21] for a general survey on the question).

Our paper is structured as follows: in Sect. 2, the gas-particles equations are specified in detail, including the collision kernel Q . Then, a non-dimensional version of those equations is provided in Sect. 3. The distributions which cancel Q are described in Sect. 4. Then, equations for the macroscopic quantities (for both phases) are written down and the system is closed (in Sect. 5). Some conclusions and perspectives are presented at the end of the paper (Sect. 6).

2 Presentation of the Inelastic Collision Kernel

2.1 General Form of the Collision Kernel

We recall here the main assumptions that we presented in the introduction of this work about the flow we consider.

We assume that the flow is constituted of a surrounding gas and of a dispersed liquid phase. This phase is itself assumed to be of relatively small volume fraction (typically between 10^{-3} and 0.2), and to be constituted of very tiny spherical incompressible droplets having all the same radius r (that is, the spray is monodisperse). The flow inside the droplets is not modeled.

As stated in the introduction, a system which models the spray under this assumption can be written down by considering the unknown $f := f(t, x, u_p, e_p) \geq 0$ for the droplets and $\rho_g := \rho_g(t, x) \in \mathbb{R}_+$, $u_g := u_g(t, x) \in \mathbb{R}^3$, $p := p(t, x) \in \mathbb{R}_+$, $E_g := E_g(t, x) \in \mathbb{R}_+$ for the gas. The set of equations is then (1)–(11), and it remains to precisely define the collision operator Q .

The assumptions that underly the establishment of this operator are the following: First, since the spray is monodisperse, no complex phenomena of coalescence or breakup of droplets are considered. For the same reason, all collisions are supposed to be binary (that is, two droplets are present at the beginning of the collision and produce two droplets at the end of the collision).

Then, since droplets are macroscopic objects, the cross section will be that of hard spheres. For the same reason, kinetic energy conservation during the process of collision is not expected in general. As a consequence, one needs to write down a model in which part of the kinetic energy is lost: models of granular media (cf. [5, 10, 37]) provide a good solution for that.

Moreover, since the internal energy of the droplets is one of the variables in f , one needs a rule to exchange internal energy during the process of collision: models for polyatomic gases (cf. [8, 13]) provide a simple solution for this physical phenomenon.

Finally, the kinetic energy which is lost has to be converted in internal energy, and to be distributed between the two outgoing droplets. Since those droplets have the same volume, we choose to divide it equally. Collecting all those ideas, we end up with a collision kernel which writes

$$\begin{aligned}
 & Q(f, f)(t, x, u_p, e_p) \\
 &= \iiint_{\substack{\sigma \in \mathbb{S}^2, u_{p_*} \in \mathbb{R}^3 \\ e_{p_*} \in \mathbb{R}^+}} \left(\frac{1}{1-a} \frac{1}{\beta^2} f(t, x, u_{p_*}, e_{p_*}) f(t, x, u_p, e_p) \right. \\
 &\quad \left. - f(t, x, u_{p_*}, e_{p_*}) f(t, x, u_p, e_p) \right) 1_{\{e_p, e_{p_*} \geq 0\}} r^2 |u_p - u_{p_*}| d\sigma du_{p_*} de_{p_*}, \quad (22)
 \end{aligned}$$

where the pre-collisional velocities u_{p_*} and u_p are defined as

$$\begin{aligned}
 u_p &= \frac{u_p + u_{p_*}}{2} - \frac{1-\beta}{4\beta} (u_p - u_{p_*}) + \frac{1+\beta}{4\beta} |u_p - u_{p_*}| \sigma, \\
 u_{p_*} &= \frac{u_p + u_{p_*}}{2} + \frac{1-\beta}{4\beta} (u_p - u_{p_*}) - \frac{1+\beta}{4\beta} |u_p - u_{p_*}| \sigma.
 \end{aligned}$$

Here, σ belongs to the unit sphere \mathbb{S}^2 , and $\int_{\sigma \in \mathbb{S}^2} d\sigma = 4\pi$. The pre-collisional internal energies e_{p*} and e_p are defined as

$$\begin{aligned} e_p &= \frac{2-a}{2-2a} e_p - \frac{a}{2-2a} e_{p*} - \frac{1}{2} \Delta E, \\ e_{p*} &= -\frac{a}{2-2a} e_p + \frac{2-a}{2-2a} e_{p*} - \frac{1}{2} \Delta E, \end{aligned}$$

where

$$\begin{aligned} \Delta E &= \frac{1}{2} (u_p'^2 + u_{p*}'^2 - u_{p*}^2 - u_p^2) \\ &= \left(\frac{1-\beta^2}{8\beta^2} \right) |u_p - u_{p*}|^2 - \frac{1-\beta^2}{8\beta} |u_p - u_{p*}| (u_p - u_{p*}) \cdot \sigma \end{aligned} \tag{23}$$

is the loss of kinetic energy (or gain of internal energy) [divided by mass].

In those formulas, $\beta := \beta(|u_p - u_{p*}|)$ is a measure of the inelasticity of the collision (the collision is elastic when $\beta = 1$), and $a := a(|u_p - u_{p*}|)$ is the parameter which measures what part of the internal energy is exchanged during a collision (no internal energy is exchanged when $a = 0$).

Note that the prefactor $\frac{1}{1-a\beta^2}$ is related to the Jacobian of the pre-collisional transform $(u_p, e_p, u_{p*}, e_{p*}) \mapsto (u_p', e_p', u_{p*}', e_{p*}')$, and to the cross section of hard spheres [38]. The model presented here is strongly reminiscent of models appearing in granular gases. The only difference is the treatment of the internal energy of the droplets.

2.2 Weak Form of the Collision Kernel and Conservations

The following weak form of the collision kernel can be obtained (at the formal level) (see [5, 10, 37]): For all test function $\Psi := \Psi(u_p, e_p)$,

$$\begin{aligned} &\iint_{u_p, e_p} Q(f, f) \Psi(u_p, e_p) du_p de_p \\ &= \iiint \iiint_{u_p, e_p, u_{p*}, e_{p*}, \sigma} f f_* [\Psi' - \Psi] r^2 |u_p - u_{p*}^*| du_p de_p du_{p*} de_{p*} \\ &= \frac{1}{2} \iiint \iiint_{u_p, e_p, u_{p*}, e_{p*}, \sigma} f f_* [\Psi_* + \Psi' - \Psi_* - \Psi] r^2 |u_p - u_{p*}^*| du_p de_p du_{p*} de_{p*}, \end{aligned} \tag{24}$$

with the following formulas for the post-collisional velocities and internal energies:

$$\begin{aligned} u_p' &= \frac{u_p + u_{p*}}{2} + \frac{1-\beta}{4} (u_p - u_{p*}) - \frac{1+\beta}{4} |u_p - u_{p*}| \sigma, \\ u_{p*}' &= \frac{u_p + u_{p*}}{2} - \frac{1-\beta}{4} (u_p - u_{p*}) + \frac{1+\beta}{4} |u_p - u_{p*}| \sigma, \\ e_p' &= \frac{2-a}{2} e_p + \frac{a}{2} e_{p*} + \frac{1}{2} \Delta E, \quad e_{p*}' = \frac{a}{2} e_p + \frac{2-a}{2} e_{p*} + \frac{1}{2} \Delta E, \end{aligned}$$

$$\begin{aligned} \Delta E &= \frac{1}{2}(u_p'^2 + u_{p_*}'^2 - u_{p_*}^2 - u_p^2) \\ &= \frac{1 - \beta^2}{8} |u_p - u_{p_*}|^2 - \frac{1 - \beta^2}{8} |u_p - u_{p_*}| (u_p - u_{p_*}) \cdot \sigma \end{aligned}$$

(remember that $\Psi_*' := \Psi(t, x, u_{p_*}', e_{p_*}')$, etc.). Specializing the weak formulation to the functions $\Psi(u_p, e_p) = m_p$, $\Psi(u_p, e_p) = m_p u_p$ and $\Psi(u_p, e_p) = \frac{1}{2} m_p u_p^2 + m_p e_p$, we get the conservations of mass, momentum and total energy :

$$\iint_{u_p, e_p} Q(f, f)(u_p, e_p) m_p du_p de_p = 0, \tag{25}$$

$$\iint_{u_p, e_p} Q(f, f)(u_p, e_p) m_p u_p du_p de_p = 0, \tag{26}$$

$$\iint_{u_p, e_p} Q(f, f)(u_p, e_p) \left[\frac{1}{2} m_p u_p^2 + m_p e_p \right] du_p de_p = 0. \tag{27}$$

2.3 Formulas for the Inelasticity and Energy Exchange Parameters

It remains to give a formula for the parameters a and β which appear in our model of collisions. This is done in this subsection, under the assumption that viscosity is the main reason why kinetic energy is lost during collisions. A more complete description of the procedure together with numerical values in a typical experiment (in the context of the nuclear industry) can be found in [29].

2.3.1 Probability Laws of Exchange of Internal Energy

We first explain the process of transfer of internal energy during the collisions. We obtain a hint of the expression of a .

A collision of droplets can be considered as a coalescence of two particles which split after some time. The droplets remain stuck together for a while, and they exchange internal energy through thermal conduction.

The time $\Delta \tau_{coll}$ of collision between two droplets p (with velocity u_p , internal energy e_p and temperature T_p) and p_* (with velocity u_{p_*} , internal energy e_{p_*} and temperature T_{p_*}) is generally assessed as $\Delta \tau_{coll} = \frac{2r}{|u_p - u_{p_*}|}$ (see [11, 23]).

Because of heat transfers, an amount $4\pi r \lambda_p (T_p - T_{p_*})$ of internal energy is exchanged by unit of time (with λ_p the thermal conductivity of the droplets, supposed to be constant) from p towards p_* . The 4π factor might not be the good geometric factor because one particle does not surround the other during the collision as the gas surrounds a particle, but this quantity gives at least the right order of magnitude. Hence, we get

$$\frac{d}{dt} (e_{p_*} - e_p) = \frac{4\pi r \lambda_p}{m_p} (T_p - T_{p_*}).$$

Supposing that the specific heat C_p of the liquid is a constant, we end up with the following evolution for the difference of internal energy between the two droplets during the collision:

$$(e_{p_*} - e_p)(t) = \exp\left(-\frac{4\pi r \lambda_p t}{C_p m_p}\right) \times (e_{p_*} - e_p)(0).$$

Using the last equality for $t = \Delta\tau_{coll}$, one gets

$$a = 1 - \exp\left(-\frac{4\pi r \lambda_p}{C_p m_p} \frac{2r}{|u_p - u_{p*}|}\right). \tag{28}$$

2.3.2 Estimate of the Inelasticity Parameter

We assess the effect of inelasticity through some computations using the T.A.B. (Taylor Analogy Break-up) model used in the Kiva code (see [4] and [2]), under the assumption that viscosity is the main factor of loss of kinetic energy during collisions between liquid droplets ([25] and [39]).

More precisely, the distortion of sphericity y satisfies the following ordinary differential equation (see [4]),

$$\ddot{y} = \frac{2}{3} \frac{\rho_g}{\rho_p} \frac{|u_p - u_g|^2}{r^2} - \frac{8\sigma_T}{\rho_p r^3} y - \frac{10\mu_p}{\rho_p r^2} \dot{y},$$

where σ_T is the surface tension of the liquid constituting the droplets, and μ_p is the dynamic viscosity of this same liquid. Assuming that the surface tension plays a negligible role (w.r.t. the viscosity) in the loss of kinetic energy during a collision, we find a characteristic time τ_c (viscous time):

$$\tau_c = 1 / \left(\frac{10\mu_p}{\rho_p r^2} \right).$$

Since the oscillatory energy is proportional to \dot{y}^2 , we see that the kinetic energy (in the center of mass) of the two droplets $E(t)$ (at a time t after the beginning of a collision) is controlled through the following exponential evolution:

$$E(t) = E(0) \exp\left(-\frac{2t}{\tau_c}\right) = \frac{1}{4} |u_p - u_{p*}|^2 \exp\left(-\frac{2t}{\tau_c}\right),$$

so that the loss of kinetic energy ΔE during a collision is

$$\Delta E = \frac{1}{4} |u_p - u_{p*}|^2 \left(1 - \exp\left(-\frac{2\Delta\tau_{coll}}{\tau_c}\right) \right).$$

Since we also have defined ΔE through

$$\begin{aligned} \Delta E &= \frac{1}{2} (u_p'^2 + u_{p*}'^2 - u_{p*}^2 - u_p^2) \\ &= \frac{1 - \beta^2}{8} |u_p - u_{p*}|^2 - \frac{1 - \beta^2}{8} |u_p - u_{p*}| (u_p - u_{p*}) \cdot \sigma, \end{aligned}$$

the loss is (at worst)

$$\Delta E = \frac{1 - \beta^2}{4} |u_p - u_{p*}|^2.$$

We end up with

$$\beta = \exp\left(-\frac{\Delta\tau_{coll}}{\tau_c}\right) = \exp\left(-\frac{20\mu_p}{\rho_p r |u_p - u_{p*}|}\right). \tag{29}$$

This procedure of course only provides a rough order of magnitude for β . This is nevertheless enough to determine if the regime of collision is elastic or not.

The equations for thick sprays being now complete, we introduce in next section a scaling based on the dimensional analysis of those equations.

3 Non-Dimensional Form of the Vlasov-Boltzmann Equation

We write down in this short section the dimensional analysis which enables to obtain a formal limit for the Vlasov-Boltzmann equation (4). In order to do so, we first introduce the following time/space typical quantities:

- t_g : typical time of the experiment,
- L : typical length of the experiment.

Next, we introduce quantities related to the gas and the droplets (remember that r, m_p, ρ_p are the radius, mass and density of droplets, and that D, Φ are the coefficients for drag force and thermal exchanges)

- N : typical number of droplets of the experiment.
- V : typical mean velocity of the droplets. We shall assume that it is also the typical thermal velocity of the droplets [that is, the square root of the variance of the velocity distribution], and the typical velocity of the gas. One has $V t_g = L$.
- I_p : typical internal energy of the droplets per mass unit.
- I_g : typical internal energy of the gas per mass unit.
- T_T : typical temperature of the droplets. We shall assume that it is also the typical temperature of the gas.
- P : Typical pressure of the gas.
- $P' = \rho_p V^2$: this quantity has the dimension of a pressure.

It is customary to introduce at this level the mean free path $\sigma = \frac{L^3}{r^2 N}$. Finally, we denote by ε the Knudsen number $\varepsilon = \frac{\sigma}{L}$. This quantity is at the basis of the passage from Boltzmann equation towards Euler equation (see [11] and [9]).

The orders of magnitude of the terms appearing in (4) are then given by the following formulas:

- Time derivative term:

$$\partial_t f \sim \frac{N}{I_p V^3 L^3 t_g} f. \tag{30}$$

- Transport term:

$$\nabla_x \cdot (f u_p) \sim \frac{N}{I_p V^3 L^3} \frac{V}{L} f \sim \partial_t f. \tag{31}$$

- Pressure term:

$$\nabla_{u_p} \cdot \left(\frac{\nabla_x P}{\rho_p} f \right) \sim \frac{N}{I_p V^3 L^3} \frac{V}{L} \frac{P}{P'} f \sim \frac{P}{P'} \partial_t f. \tag{32}$$

- Drag force term:

$$\nabla_{u_p} \cdot \left(\frac{D}{m_p} (u_p - u_g) f \right) \sim \frac{N}{I_p V^3 L^3} \frac{D}{r^3 \rho_p} \sim \left(\frac{D t_g}{r^3 \rho_p} \right) \partial_t f. \tag{33}$$

- Energy exchange term:

$$\partial_{e_p} \left(\frac{\Phi}{m_p} (T_g - T_p) f \right) \sim \frac{N}{I_p V^3 L^3} \frac{\Phi T_T}{r^3 \rho_p I_p} \sim \frac{\Phi T_T t_g}{r^3 \rho_p I_p} \partial_t f. \tag{34}$$

- Collisional term:

$$Q(f, f) \sim r^2 V I_p V^3 \left(\frac{N}{I_p V^3 L^3} \right)^2$$

(we recall that we use the hard sphere cross section)

$$\sim \frac{1}{\varepsilon} \partial_t f. \tag{35}$$

We now introduce non-dimensional quantities (denoted with a tilde) for the unknowns and parameters entering (4). That is, we consider

$$\tilde{t} = \frac{t}{t_g}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{u}_p = \frac{u_p}{V}, \quad \tilde{e}_p = \frac{e_p}{I_p}, \quad \tilde{T}_p = \frac{T_p}{T_T},$$

$$\tilde{f}(\tilde{t}, \tilde{x}, \tilde{u}_p, \tilde{e}_p) = \frac{I_p L^3 V^3}{N} f(t_g \tilde{t}, L \tilde{x}, V \tilde{u}_p, I_p \tilde{e}_p),$$

for the particles and

$$\tilde{u}_g(\tilde{t}, \tilde{x}) = \frac{u_g(t_g \tilde{t}, L \tilde{x})}{V}, \quad \tilde{T}_g(\tilde{t}, \tilde{x}) = \frac{T_g(t_g \tilde{t}, L \tilde{x})}{T_T},$$

$$\tilde{e}_g(\tilde{t}, \tilde{x}) = \frac{e_g(t_g \tilde{t}, L \tilde{x})}{I_g}, \quad \tilde{P}(\tilde{t}, \tilde{x}) = \frac{p(t_g \tilde{t}, L \tilde{x})}{P}$$

for the gas.

The equation satisfied by \tilde{f} then becomes

$$\partial_{\tilde{t}} \tilde{f} + \tilde{u}_p \cdot \nabla_{\tilde{x}} \tilde{f} + \nabla_{\tilde{x}} \cdot (\tilde{f} \tilde{\Gamma}) + \partial_{\tilde{e}_p} (\tilde{f} \tilde{\Phi}) = \frac{1}{\varepsilon} Q(\tilde{f}, \tilde{f}), \tag{36}$$

where

$$\tilde{\Gamma} = \frac{P}{P'} \nabla_{\tilde{x}} \tilde{P} + C_2 (\tilde{u}_p - \tilde{u}_g),$$

$$\tilde{\Phi} = C_3 (\tilde{T}_g - \tilde{T}_p),$$

$$C_2 = \frac{D}{m_p} t_g,$$

$$C_3 = \frac{\Phi T_T t_g}{m_p I_p}.$$

We shall now study the limit of (36) when $\varepsilon \rightarrow 0$. We see that this limit makes sense when the typical parameters of the experiment under study are such that

$$\text{from (30) and (31), } 1 \gg \varepsilon, \tag{37a}$$

$$\text{from (32), } \frac{P'}{P} \sim 1, \tag{37b}$$

$$\text{from (33), } \frac{m_p}{Dt_g} \sim 1, \tag{37c}$$

$$\text{from (34), } \frac{m_p I_p}{\Phi T_T t_g} \sim 1. \tag{37d}$$

A typical situation appearing in the nuclear industry where those assumptions are fulfilled is described in [29]. Note that other scalings, based on a different ordering of the various terms appearing in (36) can be performed. In the context where collisions are replaced by a diffusion operator (in the u_p variable), we wish to point out two such scalings, performed in [19] and [20]. One of the main differences with our own work is the fact that in those papers, the drag force is of order ε^{-1} . In our setting, the drag force is of order 1 because in the application to the nuclear industry that we have in mind (cf. [29]), the spray is produced by the introduction of “not too small” droplets of liquid tin in a gas (air) at a “not too big” relative velocity.

4 Limit of the PDF in the Scaling

In order to pass to the limit (at the formal level) in (36) when $\varepsilon \rightarrow 0$, we study the solutions of the functional equation $Q(f, f) = 0$, when collisions are truly inelastic, that is when $\beta := \beta(|u_p - u_{p*}|) \in [0, 1[$. The computation of the exchange of kinetic energy leads to

$$\begin{aligned} & \iint_{u_p, e_p} Q(f, f) \frac{1}{2} m_p u_p^2 du_p de_p \\ &= \iiint\limits_{u_p, e_p, u_{p*}, e_{p*}} \frac{1 - \beta^2}{8} f f_* 4\pi r^2 m_p |u_p - u_{p*}|^3 du_{p*} de_{p*} du_p de_p, \end{aligned} \tag{38}$$

so that the effect of inelastic collisions is to concentrate the velocities of the droplets. Note first that when considering only the evolution of velocities, in absence of internal energy exchange, the convergence towards a Dirac mass is rigorously proven in [5], [37] or [17] for solutions of the spatially homogeneous Boltzmann equation $\partial_t f = Q(f, f)$ (for β constant).

We now wish to show, at the formal level, that when considering the evolution of both velocities and internal energies in $\partial_t f = Q(f, f)$,

$$\lim_{t \rightarrow +\infty} f(t, u_p, e_p) = G \delta_{u_p=v}(u_p) \otimes \delta_{e_p=e}(e_p), \tag{39}$$

with $v \in \mathbb{R}^3, G \geq 0, e > 0$.

Note that this cannot be done directly by the study of the solutions of $Q(f, f) = 0$ since all densities of the form

$$f(u_p, e_p) = \delta_{u_p=v}(u_p) \otimes \mu(e_p), \tag{40}$$

where μ is a positive measure, are such solutions (equilibria).

The explanation of what at first glance seems an inconsistency between (39) and (40) is the following: when μ is not a Dirac mass, the equilibria of the form $\delta_{u_p=v}(u_p) \otimes \mu(e_p)$ are unstable. More precisely, as soon as the density f is not exactly a Dirac mass w.r.t. the variable u_p , some collisions occur and consequently some energy will be exchanged between the droplets, so that μ will converge towards a Dirac mass. Note that other instabilities in the context of granular media have been studied. We refer in particular to [5] for the instability of the constant states (as far as mass and velocity are concerned; the temperature behaving like t^{-2}) w.r.t. large waves perturbations for solutions of dissipative Euler systems.

4.1 The Case of Constant Coefficients of Inelasticity and Energy Exchange

We assume in the following computation that a and β are constant, for the sake of simplicity. An extension of this computation in a case in which a and β are not constant is given in Sect. 4.2.

In order to do so, we first recall Haff’s law [22]: For $f := f(t, u_p)$ satisfying the spatially homogeneous equation $\partial_t f = Q(f, f)$ (with $\beta \in [0, 1[$ and no exchange of energy involved), the following estimate holds:

$$\frac{m}{1+t^2} \leq T(t) \leq \frac{M}{1+t^2}, \tag{41}$$

where

$$T(t) := \frac{\int_{u_p} f(t, u_p) \frac{1}{3} m_p (u_p - v)^2 du_p}{\int_{u_p} f(t, u_p) m_p du_p},$$

and where m and M are constants depending on initial data. A rigorous proof of this result can be found in [30] and [31] (when β is a constant).

A first hint of the proof can be found in [5] when one assumes that $|v - v_*|$ is replaced by a term proportional to \sqrt{T} . Our goal here is to estimate the evolution of the mean internal energy along the solutions of the equation

$$\partial_t f(t, u_p, e_p) = Q(f, f)(t, u_p, e_p). \tag{42}$$

The computations that we provide are only approximations. They give an idea of what should be the evolution of the quantity

$$g(t) := \frac{\iint_{u_p, e_p} f(t, u_p, e_p) m_p (e_p - e(t))^2 du_p de_p}{\iint_{u_p, e_p} f(t, u_p, e_p) m_p du_p de_p},$$

that is the variance of f w.r.t. e_p . They will be sustained in next subsection by numerical simulations. Note first that thanks to the conservation of mass,

$$\begin{aligned}
 g'(t) &= \frac{\iint Q(f, f)(t, u_p, e_p)m_p(e_p - e(t))^2 du_p de_p}{\iint f(t, u_p, e_p)m_p du_p de_p} \\
 &= \left(-a \left(1 - \frac{a}{2} \right) \iiint \frac{1}{2} f f^* 4\pi r^2 (e_p - e_{p_*}) |u_p - u_{p_*}| du_p du_{p_*} de_p de_{p_*} \right. \\
 &\quad \left. + \frac{1}{2} \iiint 4\pi r^2 f f^* \left[\frac{1}{2} \Delta E^2 + \Delta E (e_p + e_{p_*} - 2e) \right] \right. \\
 &\quad \left. \times |u_p - u_{p_*}| du_p du_{p_*} de_p de_{p_*} \right) / \iint f du_p de_p. \tag{43}
 \end{aligned}$$

We use the following approximation based on Haff’s law: in all computations we replace $|u_p - u_{p_*}|$ by $\sqrt{6T}$ (the 6 comes from the fact that we are in 3D): it is more or less the same approach as in [5]. Then,

$$\begin{aligned}
 g'(t) &\sim \left(-a \left(1 - \frac{a}{2} \right) \sqrt{6T} \iiint \frac{1}{2} f f^* 4\pi r^2 (e_p - e_{p_*})^2 du_p du_{p_*} de_p de_{p_*} \right. \\
 &\quad \left. + \iiint 2\pi r^2 f f^* \left(\frac{1}{2} \left(\frac{1 - \beta^2}{4} 6T \right)^2 + \frac{1 - \beta^2}{4} 6T (e_p + e_{p_*} - 2e) \right) \right. \\
 &\quad \left. \times \sqrt{6T} du_p du_{p_*} de_p de_{p_*} \right) / \iint_{u_p, e_p} f du_p de_p. \tag{44}
 \end{aligned}$$

We use the identities

$$\begin{aligned}
 &\iiint f f^* (e_p - e_{p_*})^2 du_p du_{p_*} de_p de_{p_*} \\
 &= \iiint f f^* [(e_p - e)^2 + (e - e_{p_*})^2] du_p du_{p_*} de_p de_{p_*} \\
 &= 2g(t) \left(\iint_{u_p, e_p} f du_p de_p \right)^2, \tag{45}
 \end{aligned}$$

and

$$\iiint f f^* (e_p + e_{p_*} - 2e) du_p du_{p_*} de_p de_{p_*} = 0. \tag{46}$$

As a consequence, we obtain the following (approximate) ODE for the evolution of g :

$$g'(t) \sim (1 - \alpha) \left(-a \left(1 - \frac{a}{2} \right) \sqrt{6} \frac{3}{r} T(t)^{1/2} g(t) + \frac{1}{4} \left(\frac{1 - \beta^2}{4} \right)^2 \frac{3}{r} (6T(t))^{5/2} \right), \tag{47}$$

where

$$1 - \alpha = \iint_{u_p, e_p} f \frac{4}{3} \pi r^3 du_p de_p$$

is the volume occupied by the droplets (in the homogeneous setting that we consider, this quantity is a constant). Using now (according to Haff’s law) the approximation $T(t) =$

$\frac{c_1^2}{(1+c_2t)^2}$ where c_1 and $c_2 > 0$, we solve (47) and obtain (except in the exceptional case when $\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{6a}(1-a/2) = 4$):

$$\begin{aligned}
 g(t) \sim & \frac{g(0)}{(1+c_2t)^{\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{6a}(1-a/2)}} \\
 & + \frac{1}{4} \left(\frac{1-\beta^2}{4} \right)^2 \frac{(\sqrt{6}c_1)^5/c_2}{\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{6a}(1-a/2) - 4} \\
 & \times \left[(1+c_2t)^{-4} - (1+c_2t)^{-\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{6a}(1-a/2)} \right]. \tag{48}
 \end{aligned}$$

We now discuss the behavior of g according to the sign of $\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{6a}(1-a/2) - 4$.

- When $4 < \frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{6a}(1-a/2)$: we get

$$g(t) \sim \frac{Cst}{(1+c_2t)^4}. \tag{49}$$

This is the situation when thermal exchanges are predominant: $\sqrt{g(t)}$ then converges to zero as rapidly as the temperature $T(t)$ (note that \sqrt{g} has the same dimension as an energy).

- When $4 > \frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{6a}(1-a/2)$, we get

$$g(t) \sim \frac{Cst}{(1+c_2t)^{\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{6a}(1-a/2)}},$$

so that $\sqrt{g(t)}$ still converges towards 0, but this convergence is slower than that of the temperature $T(t)$. It can even be very slow when a is close to 0 (that is, when the exchanges of internal energy are of small amplitude).

Note finally that the exceptional case $\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{6a}(1-a/2) = 4$ leads to a formula close to (49) [but with a logarithmic correction].

The previous computations show (though not rigorously) that the only stable equilibrium of $\partial_t f = Q(f, f)$ in the case of inelastic collisions ($\beta \in [0, 1]$) are functions defined by (39).

We now present a numerical simulation which confirms the approximate computations presented above. We present some numerical tests for the spatially homogeneous Boltzmann equation $\partial_t f = Q(f, f)$, when Q is the inelastic collision kernel defined by (22), with a and β fixed constants. The computations are performed thanks to a particle method (cf. [4, 34]), where the density $f := f(t, u_p, e_p)$ is approximated by a sum of Dirac masses with the same numerical weight (that is, $f(t, u_p, e_p) \sim w \sum_{i=1}^N \delta_{u_p=u_{p_i}; e_p=e_{p_i}}$). This set of numerical particles then evolves according to Bird’s method (cf. [7]). The tests which are presented correspond to the following parameters:

$$r = 10^{-4}, \quad f(0, u_p, e_p) = Cst \mathbf{1}_{u_p \in [-10^4, 10^4]^3; e_p \in [5 \cdot 10^5, 5 \cdot 10^6]}.$$

About 10^4 numerical particles are used.

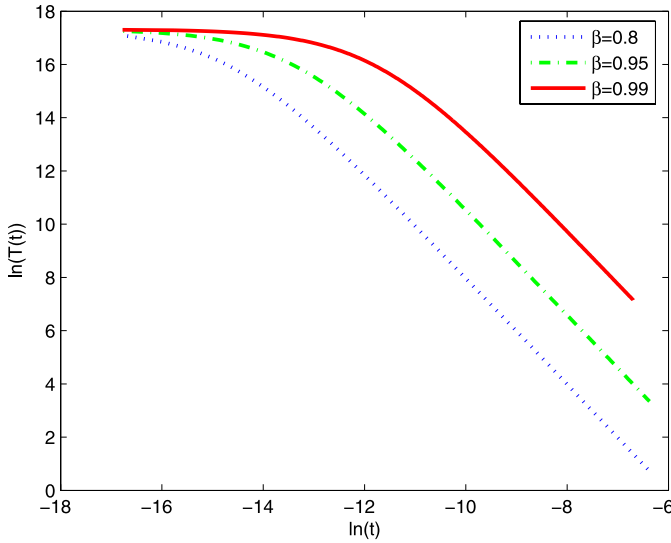


Fig. 1 Behavior of kinetic temperature: $\ln T$ as a function of $\ln t$ for different β

First Test: Convergence Towards the Dirac Mass w.r.t. Velocity; Haff's Law We check that Haff's law holds for $a = 1$ and $\beta = 0.99, 0.95, 0.8$: we plot the results in logarithm scale: we expect to get a (asymptotically) straight line whose slope is -2 (since Haff's law means that $T(t) \sim t^{-2}$).

It is indeed what we observe in Fig. 1. Note also that, as expected, the convergence is slower when β increases.

Second Test: Convergence Towards the Dirac Mass w.r.t. the Internal Energy We now check the convergence towards the Dirac mass w.r.t. internal energy. We fix $\beta = 0.99$ and let a vary between 0.01 and 1.0. We plot

$$W = \ln \left(\frac{\iint f(t, u_p, e_p) |e_p - e(t)| de_p du_p}{\iint f(t, u_p, e_p) de_p du_p} \right)$$

as a function of $\ln(t)$.

As can be seen in Fig. 2, the more a increases, the more the coefficients of the asymptotic straight line tend to -2 . More precisely (in accordance with the theoretical computation), we see that there exists a critical a (around 0.06) which separates a zone in which the behavior of W seems to be in t^{-2} , and a zone in which it is rather in $t^{-\theta}$, with $\theta \in]0, 2[$ depending on a . Finally, we observe that for small a , the function W increases during a certain amount of time: thermal exchanges are then not significant enough to completely counterbalance the positive term in (47) (that is, the transfer of kinetic energy to internal energy) at all times.

4.2 The Case when the Coefficients of Inelasticity and Internal Energy Exchange Depend on Relative Velocity

We (briefly) present here a formal computation showing that the results of Sect. 4.1 obtained when a and β are constant still hold when a and β are given by (28) and (29). The computation performed in Sect. 4.1 for the evolution of temperature (with the approximation

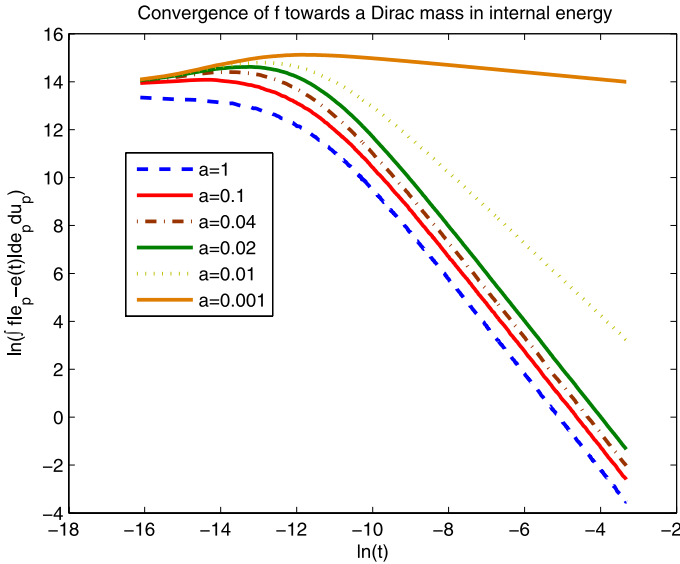


Fig. 2 Convergence in internal energy: $\ln(\int f(t, u_p, e_p)|e_p - e(t)| de_p du_p / \int f(t, u_p, e_p) de_p du_p)$ as a function of $\ln t$ for various values of parameters

$|u_p - u_{p*}| \sim \sqrt{6T}$) leads to the following ODE, when β is given by formula (29):

$$\begin{aligned} \frac{d}{dt} T &\sim -\frac{C}{r} \left(1 - \exp\left(-2\frac{\Delta\tau_{coll}}{\tau_c}\right) \right) (1 - \alpha) T^{\frac{3}{2}} \\ &\sim -\frac{C}{r} \left(1 - \exp\left(-\frac{4r}{\sqrt{6T}} \frac{1}{\tau_c}\right) \right) (1 - \alpha) T^{\frac{3}{2}}, \end{aligned} \tag{50}$$

for some $C > 0$. When T is large, $\frac{d}{dt} T \sim -\frac{4C}{\tau_c} (1 - \alpha) T$ so that T decays exponentially. When T becomes small enough, T satisfies $\frac{d}{dt} T \sim -\frac{C}{r} (1 - \alpha) T^{\frac{3}{2}}$ and we are back to the situation that we studied in Sect. 4.1 with β constant (and close to 0). Similar computations can be done for the quantity g . We now present a numerical simulation which sustains those computations.

We use the expression of a and β found in Sects. 2.3.1 and 2.3.2. The numerical code is similar to the code used previously. As one can see in Fig. 3, the temperature $T(t)$ does converge to zero. At the beginning, this convergence is exponential, whereas as soon as T becomes small enough, the rate of convergence corresponds to Haff’s law, i.e. is in $\frac{1}{T^2}$. Next we observe the convergence of the internal energy in Fig. 4. We see that the internal energy indeed converges to a Dirac mass in this simulation.

5 Fluid of Particles

According to the dimensional analysis of Sect. 3, we end up with the following set of scaled equations for the spray:

$$\partial_t (\alpha^\varepsilon \rho_g^\varepsilon) + \nabla_x \cdot (\alpha^\varepsilon \rho_g^\varepsilon u_g^\varepsilon) = 0, \tag{51}$$

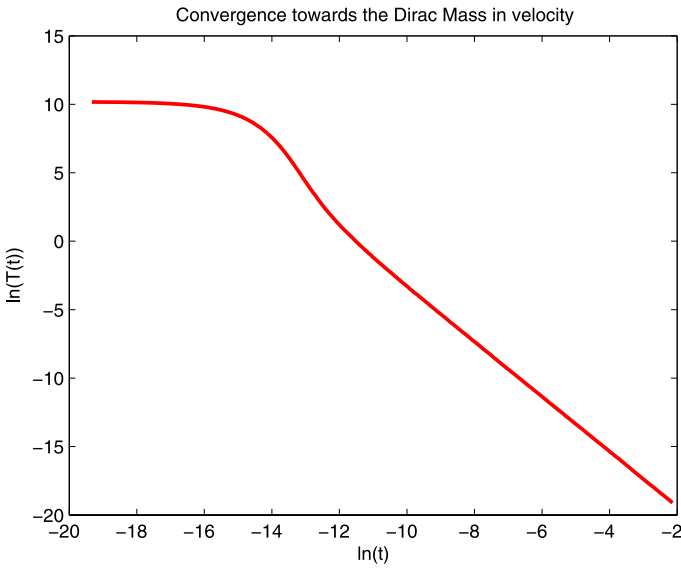


Fig. 3 Convergence towards the Dirac mass in velocity: $\ln T$ as a function of $\ln t$

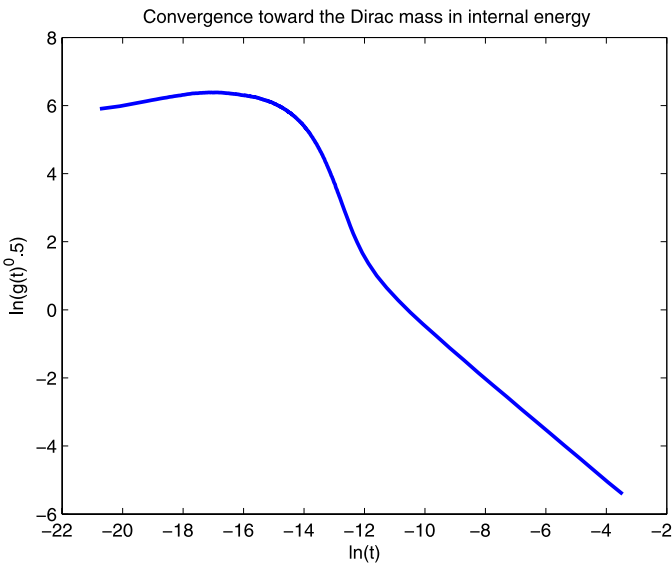


Fig. 4 Convergence towards the Dirac mass in internal energy: $\ln(\iint f(t, u_p, e_p)|e_p - e(t)|de_p du_p / \iint f(t, u_p, e_p)de_p du_p)$ as a function of $\ln t$

$$\partial_t (\alpha^\varepsilon \rho_g^\varepsilon u_g^\varepsilon) + \nabla_x \cdot (\alpha^\varepsilon \rho_g^\varepsilon u_g^\varepsilon \otimes u_g^\varepsilon) + \nabla_x p^\varepsilon = -A^\varepsilon, \tag{52}$$

$$\partial_t (\alpha \rho_g^\varepsilon E_g^\varepsilon) + \nabla_x \cdot \left(\alpha^\varepsilon \rho_g^\varepsilon \left(E_g^\varepsilon + \frac{p^\varepsilon}{\rho_g^\varepsilon} \right) u_g^\varepsilon \right) + p^\varepsilon \partial_t \alpha^\varepsilon = -B_1^\varepsilon - B_2^\varepsilon, \tag{53}$$

$$\partial_t f^\varepsilon + u_p^\varepsilon \cdot \nabla_x f^\varepsilon + \nabla_{u_p} \cdot (f^\varepsilon \Gamma^\varepsilon) + \partial_{e_p} (f^\varepsilon \phi^\varepsilon) = \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon), \tag{54}$$

where

$$m_p \Gamma^\varepsilon = -\frac{m_p}{\rho_p} \nabla_x P^\varepsilon - D(u_p - u_g^\varepsilon); \quad m_p \phi^\varepsilon = \Phi(T_g^\varepsilon - T_p), \tag{55}$$

$$A^\varepsilon = \iint_{u_p, e_p} m_p \Gamma^\varepsilon f^\varepsilon du_p de_p, \tag{56}$$

$$B_1^\varepsilon = \iint_{u_p, e_p} m_p \left(\Gamma^\varepsilon + \frac{\nabla_x P^\varepsilon}{\rho_p} \right) \cdot u_p f^\varepsilon du_p de_p, \tag{57}$$

$$B_2^\varepsilon = \iint_{u_p, e_p} m_p \phi^\varepsilon f^\varepsilon du_p de_p. \tag{58}$$

In this section, we present the computations which enable to pass to the limit at the formal level in (51)–(58), when $\varepsilon \rightarrow 0$. These formal computations are based on the same principle as the traditional passage from the Boltzmann eq. towards fluid mechanics: we first take moments of (54), and then close the corresponding equations thanks to the study (in Sect. 4) of the solutions of $Q(f, f) = 0$ (more precisely, of the large time behavior of the solutions of the spatially homogeneous equation $\partial_t f = Q(f, f)$).

We define the following quantities associated with the moments of order zero (mass), one (momentum), two (energy, pressure (Reynolds’) tensor) and three (flux of energy) of the fluid of particles (the notations used here are coherent with those of Sect. 1):

$$\begin{aligned} (1 - \alpha)\rho &= \iint_{u_p, e_p} f m_p du_p de_p, & (1 - \alpha)\rho v &= \iint_{u_p, e_p} f m_p u_p du_p de_p, \\ (1 - \alpha)\rho e_c &= \iint_{u_p, e_p} \frac{1}{2} f m_p |u_p|^2 du_p de_p, & (1 - \alpha)\rho e &= \iint_{u_p, e_p} f m_p e_p du_p de_p, \\ (1 - \alpha)\rho E &= \iint_{u_p, e_p} f \left\{ \frac{1}{2} m_p |u_p|^2 + m_p e_p \right\} du_p de_p, \\ (1 - \alpha)P' &= \iint_{u_p, e_p} f m_p (v - u_p) \otimes (v - u_p) du_p de_p, \\ (1 - \alpha)q &= \iint_{u_p, e_p} f m_p (v - u_p)^2 (u_p - v) du_p de_p. \end{aligned}$$

Note that the pressure tensor P' will appear in our set of equations because the fluid of droplets does not “see” the same pressure as the gas. This extra term of pressure, sometimes called interfacial pressure, appears (usually in a non tensorial form) in many works concerned with the modeling of two-phase flows (see [36] and [18] for example). This pressure tensor vanishes when all the droplets have the same velocity (in the limit $\varepsilon \rightarrow 0$).

We now integrate the Boltzmann equation against $m_p du_p de_p$ (mass conservation), $m_p u_p du_p de_p$ (momentum conservation), and $m_p [\frac{1}{2}|u_p|^2 + e_p] du_p de_p$ (energy conservation). We use properties (25), (26) and (27) of the collision kernel. This leads to

$$\partial_t ((1 - \alpha)\rho) + \nabla_x \cdot ((1 - \alpha)\rho v) = 0$$

(remember that $\rho = \rho_p$ is a constant),

$$\begin{aligned} & \partial_t((1 - \alpha)\rho v) + \nabla_x \cdot ((1 - \alpha)\rho v \otimes v) + (1 - \alpha)\nabla_x p + \nabla_x \cdot ((1 - \alpha)P') \\ &= - \iint_{u_p, e_p} D(u_p - u_g) f du_p de_p, \end{aligned}$$

and

$$\begin{aligned} & \partial_t((1 - \alpha)\rho E) + \nabla_x \cdot \left((1 - \alpha)\rho \left(E + \frac{p}{\rho} \right) v \right) + p \partial_t(1 - \alpha) \\ &+ \nabla_x \cdot ((1 - \alpha)(P'v + q)) \\ &= - \iint_{u_p, e_p} D(u_p - u_g) \cdot u_p f du_p de_p + \iint_{u_p, e_p} \Phi(T_g - T_p) f du_p de_p. \end{aligned} \tag{59}$$

We now close the equations by formally letting ε go to 0 in (51)–(58). According to the results of Sect. 4, we know (at the formal level) that $f^\varepsilon \rightarrow f$, with

$$f(t, x, u_p, e_p) = G(t, x) \delta_{u_p=v(t,x)}(u_p) \delta_{e_p=e(t,x)}(e_p). \tag{60}$$

We end up with a system of 6 equations which write (remember that $e_g = E_g - \frac{1}{2}u_g^2$ and $e = E - \frac{1}{2}v^2$):

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \tag{61}$$

$$\partial_t((1 - \alpha)\rho) + \nabla_x \cdot ((1 - \alpha)\rho v) = 0, \tag{62}$$

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \alpha \nabla_x p = -\tilde{A}, \tag{63}$$

$$\partial_t((1 - \alpha)\rho v) + \nabla_x \cdot ((1 - \alpha)\rho v \otimes v) + (1 - \alpha)\nabla_x p = \tilde{A}, \tag{64}$$

$$\partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p \partial_t \alpha = -\tilde{B}_1 - \tilde{B}_2, \tag{65}$$

$$\partial_t((1 - \alpha)\rho E) + \nabla_x \cdot \left((1 - \alpha)\rho \left(E + \frac{p}{\rho} \right) v \right) + p \partial_t(1 - \alpha) = \tilde{B}_1 + \tilde{B}_2, \tag{66}$$

where \tilde{A} , \tilde{B}_1 and \tilde{B}_2 are defined in the introduction, the functions \tilde{D} and $\tilde{\Phi}$ being the same as D , Φ , but taken at points v, e instead of u_p, e_p . We recall the equations of state which complete this system:

$$p = P_1(\rho_g, e_g), \quad T_g = T_1(\rho_g, e_g), \tag{67}$$

$$\rho = \rho_p, \quad T = T_2(e). \tag{68}$$

At the end, we obtain the announced link between the set of Eulerian-Lagrangian equations and the set of Eulerian-Eulerian equations.

Remark When collisions are elastic (that is, $\beta = 1$), the computation runs thus: first, $f^\varepsilon \rightarrow f$ (formally), with

$$f(t, x, u_p, e_p) = \frac{Z(t, x)}{(2\pi T(t, x))^{\frac{3}{2}}} \exp\left(-\frac{(u_p - v(t, x))^2}{2T(t, x)}\right) \delta_{e_p=e(t,x)}(e_p), \tag{69}$$

where Z, v, e, T are macroscopic moments which satisfy the following closed set of seven equations for seven unknowns, that is

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \tag{70}$$

$$\partial_t((1 - \alpha)\rho) + \nabla_x \cdot ((1 - \alpha)\rho v) = 0, \tag{71}$$

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \alpha \nabla_x p = \mathcal{M}(\rho, 1 - \alpha, v, T, u_g, D), \tag{72}$$

$$\begin{aligned} \partial_t((1 - \alpha)\rho v) + \nabla_x \cdot ((1 - \alpha)\rho v \otimes v) + (1 - \alpha)\nabla_x p + \nabla_x((1 - \alpha)\rho T) \\ = -\mathcal{M}(\rho, 1 - \alpha, v, T, u_g, D), \end{aligned} \tag{73}$$

$$\begin{aligned} \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g}\right) u_g\right) + p\partial_t\alpha \\ = \mathcal{I}(\rho, 1 - \alpha, v, T, u_g, D) - \Phi(T_g - T) \frac{(1 - \alpha)\rho}{m_p}, \end{aligned} \tag{74}$$

$$\begin{aligned} \partial_t((1 - \alpha)\rho e_c) + \nabla_x \cdot ((1 - \alpha)\rho(e_c + T)v) + (1 - \alpha)v \cdot \nabla_x p \\ = -\mathcal{I}(\rho, 1 - \alpha, v, T, u_g, D), \end{aligned} \tag{75}$$

$$\begin{aligned} \partial_t((1 - \alpha)\rho e) + \nabla_x \cdot ((1 - \alpha)\rho e v) + p(\partial_t(1 - \alpha) + \nabla_x \cdot ((1 - \alpha)v)) \\ = \Phi(T_g - T) \frac{(1 - \alpha)\rho}{m_p}, \end{aligned} \tag{76}$$

where $E_g = e_g + \frac{1}{2} |u_g|^2$, $e_c = \frac{3}{2} T + \frac{1}{2} |v|^2$, and \mathcal{M}, \mathcal{I} are ‘‘almost’’ explicit functions. Their expression can be found in [29]. Remember also that the following equations of state are added to (70)–(76):

$$p = P_1(\rho_g, e_g), \quad T_g = T_1(\rho_g, e_g), \quad T_p = T_2(e), \quad \rho = \rho_p. \tag{77}$$

Note that this system is not found in textbooks about two-phase flows since usually the collisions between droplets are considered to be inelastic.

6 Conclusion and Perspectives

We now wish to briefly comment some of the issues related to this paper.

Firstly, we wish to explain what can be the extensions of the asymptotics presented in this work: the presence of (molecular or turbulent) diffusion in the gas equations does not change

the computations. It is also possible in principle to take into account chemistry terms (e.g. combustion terms) in the equations: this leads however to serious complications. Finally, it is known that polydispersion plays a decisive role in the construction of macroscopic models starting from spray equations (cf. [14]). In general, it is not possible to guess the evolution of droplets w.r.t. radius, and one has to cut into “sections” the various possible radiuses r . It however sometimes happens that processes of coagulation/breakup lead to such specific profiles (cf. for example [1]). In such (unfortunately unrealistic, at least when sprays are concerned) situations, two-phase macroscopic equations can be obtained (at the formal level) by an asymptotics.

Secondly, we would like to emphasize the extreme difficulty of making rigorous the passage to the limit that we propose (even in a “small time” setting). This is related to the very bad mathematical behavior of the limiting equations (12)–(19). Those equations are not written in conservative form and have a domain of non hyperbolicity (cf. [35]). Moreover, the set of equations (1)–(11) has not yet been studied from the mathematical point of view. It might indeed present a behavior as bad as the limiting system [though this guess is not yet sustained by convincing arguments]. One possibility could be to try to pass to the limit in an analogous system, where the molecular viscosity of the gas is not neglected (then the limiting equations are better behaved, cf. [35]).

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